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Infinite groups with large balls of torsion elements and small entropy

By

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Abstract. We exhibit infinite, solvable, virtually abelian groups with a fixed number of generators, having arbitrarily large balls consisting of torsion elements. We also provide a sequence of 3-generator non-virtually nilpotent polycyclic groups of algebraic entropy tending to zero. All these examples are obtained by taking appropriate quotients of finitely presented groups mapping onto the first Grigorchuk group.

The Burnside Problem asks whether a finitely generated group all of whose elements have finite order must be finite. We are interested in the following related question: fix n sufficiently large; given a group Γ, with a finite symmetric generating subset S such that every element in the n-ball is torsion, is Γ finite? Since the Burnside Problem has a negative answer, *a fortiori* the answer to our question is negative in general. However, it is natural to ask for it in some classes of finitely generated groups for which the Burnside Problem has a positive answer, such as linear groups or solvable groups. This motivates the following proposition, which in particular answers a question of Breuillard to the authors.

Proposition 1. *For every n, there exists a group G, generated by a 3-element subset S consisting of elements of order 2, in which the n-ball consists of torsion elements, and which satisfies one of the additional assumptions:*

(i) *G is solvable, virtually abelian, and infinite (more precisely, it has a free abelian normal subgroup of finite 2-power index); in particular it is linear.*
(ii) *G is polycyclic but not virtually nilpotent.*
(iii) *G is solvable but not polycyclic.*

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Remark 2.

(1) The groups in Proposition 1 can actually be chosen to be 2-generated: indeed, if G is generated by three involutions a, b, c, then the subgroup generated by ab and bc has index at most two.
(2) Natural stronger hypotheses are the following: being linear in fixed dimension; being solvable of given solvability length. We have no answer in these cases. It is also natural to ask what happens it we fix a torsion exponent.
(3) By [12, Corollaire 2, p. 90], if G is a group and S is any finite generating subset for which the 2-ball of G consists of torsion elements, then G has Property (FA): every action of G on a tree has a fixed point. In particular, if G is infinite, then by Stallings' Theorem [13] it cannot be virtually free.
(4) For every sufficiently large prime p, and for all n, there exists a non-elementary, 2-generated word hyperbolic group in which the n-ball consists of elements of p-torsion [9].
(5) We give more precise statements in the sequel: in (i), the free abelian subgroup can be chosen of index 2^{a_n}, where $a_n \sim 13n^k$ (that is, $a_n/(13n^k) \to 1$), where $k \cong 6.60$ is a constant (see Corollary 10).

With a similar construction, we obtain results on the growth exponent. Let G be generated by a finite symmetric set S, and denote by B_n the n-ball in G. Then, by a standard argument [7, Proposition VI.56], the limit $h(G, S) = \lim \frac{1}{n} \log(\#(B_n))$ exists. The (algebraic) entropy of G is defined as $h(G) = \inf_S h(G, S)$, where S ranges over all finite symmetric generating subsets of G. Osin has proved in [10], [11] that, for an elementary amenable finitely generated group, $h(G) = 0$ if and only if G is virtually nilpotent; on the other hand, Wilson [14] has constructed a finitely generated group with $h(G) = 0$ which is not virtually nilpotent; see [2] for a simpler example. Relying on former work by Grigorchuk [5], Osin observes in [11] that there exist elementary amenable groups (actually they are virtually solvable) with $h > 0$ arbitrary close to 0. This last result can be improved as follows.

Proposition 3. *For every $\varepsilon > 0$, there exists a polycyclic, virtually metabelian, 3-generated group G with $0 < h(G) < \varepsilon$.*

Propositions 1 and 3 are obtained by approximating the Grigorchuk group, first introduced in [4], by finitely presented groups.

We recall below the definition of a family of 3-generated groups Γ_n, which are successive quotients (Γ_{n+1} is a quotient of Γ_n for all n). These are finitely presented groups obtained by truncating a presentation of Grigorchuk's group. They are proved in [6] to be virtually direct products of nonabelian free groups; they have larger and larger balls of torsion elements, and their entropy tends to zero. We get Propositions 1 and 3 by considering appropriate solvable quotients of the groups Γ_n.

Following Lysionok [8], the first Grigorchuk group is presented as follows. We start from the 3-generated group $\Gamma_{-1} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1\rangle$. Elements u_n and v_n of Γ_{-1} are defined below. Then, for $0 \leqq n \leqq \infty$, Γ_n is defined as the quotient of Γ_{-1} by the relations u_i for $i < n + 1$ and v_i for $i < n$. The first Grigorchuk group $\Gamma = \Gamma_\infty$ has a wealth of remarkable properties. The most celebrated one is that Γ has non-polynomial subexponential growth [5]. It is also a 2-group, i.e., a group in which every element is of finite 2-power order, and is just-infinite, i.e., it is infinite but all of its proper quotients are finite.

We now construct the relators u_n and v_n. Consider the substitution σ defined by $\sigma(a) = aca$, $\sigma(b) = d$, $\sigma(c) = b$, $\sigma(d) = c$; extend its definition to words in the natural way, and finally observe that it defines a group endomorphism of Γ_{-1}. Set $u_0 = (ad)^4$, $v_0 = (adacac)^4$, $u_n = \sigma^n(u_0)$, $v_n = \sigma^n(v_0)$.

For all $n \geqq -1$, the natural homomorphism $\Gamma_n \to \mathbb{Z}/2\mathbb{Z}$ sending b, c, d to 0 and a to 1 has kernel Ξ_n of index two.

We will focus on the finitely presented groups Γ_n rather than on Γ. Individually, and up to commensurability, the structure of these groups is not of special interest: Γ_n is commensurable to a direct product of 2^n non-abelian free groups [6, Proposition 12]. However, since Γ_∞ is torsion, for all n, there exists $i(n)$ such that every element in the n-ball of $\Gamma_{i(n)}$ is torsion. A quantitative statement is given in the following proposition, whose proof appears in the appendix. Let $\lambda \cong 1.25$ be the real root of the polynomial $2X^3 - X^2 - X - 1$, and set $i(n) = \lfloor \log_\lambda(n) - 1 \rfloor$.

Proposition 4. *In the n-ball of $\Gamma_{i(n)}$ (for the word metric), every element is of $2^{i(n)+1}$-torsion.*

The following proposition, which specifies [6, Proposition 12], describes the structure of Γ_n.

Proposition 5. *For every $n \geqq 0$, Γ_n has a normal subgroup H_n of index 2^{α_n}, where $\alpha_n \leqq (13 \cdot 4^n - 1)/3$, such that H_n is a subgroup of index 2^{β_n} in a finite direct product of 2^n nonabelian free groups of rank 3, where $\beta_n \leqq (13 \cdot 4^n - 15 \cdot 2^n + 2)/3$.*

Remark 6. The main difference with [6, Proposition 12] is that the finite index subgroup they construct is not normal. Of course one could take a smaller normal subgroup of finite index, but that one need not *a priori* be of 2-power index, a fact we require to obtain solvable (and not only virtually solvable) groups in Propositions 1 and 3.

We use the following elementary lemma.

Lemma 7. *Let G be a group, and let H be a proper subgroup of index 2^a in G, normalized by a subgroup of index two in G. Let N be the intersection of all conjugates of H. Then N has index 2^b in G, for some integer $b \leqq 2a - 1$.*

Proof. If H is normal in G, the result is trivial. Otherwise, consider the unique conjugate $H' \neq H$ of H, so that $N = H \cap H'$. Taking the quotient by N, we can suppose that $H \cap H' = \{1\}$ and we are reduced to proving that G is a 2-group of order $d \leqq 2^{2a-1}$. Let W be the normalizer of H. Since it has index 2 in G, it is normal in G, so that $H' \subset W$. Since H and H' are both normal subgroups of W and $H \cap H' = \{1\}$, $[H, H'] = \{1\}$. Accordingly, HH' is a normal subgroup of G, contained in W, and is naturally the direct product of H and H'. The order of H is $d/2^a$, so that the order of HH' is $d^2/2^{2a}$, and hence the index of HH' in G is $2^{2a}/d$. This proves that d is a power of 2, and $d \leqq 2^{2a}$; actually $d \leqq 2^{2a-1}$ because HH' is contained in W, hence has index $\geqq 2$ in G. □

Remark 8. In Lemma 7, the assumption that the normalizer has index at most two is sharp: in the alternating group A_4, there are four subgroups of index 4, all conjugate; they have pairwise trivial intersection, hence of index 12, which is not a power of 2.

Recall that $\Xi_0 \subset \Gamma_0$ is a subgroup of index 2; it is generated by b, c, d, aba, aca, ada. By [6, Proposition 1], the assignment $i_0(b) = (a, c)$, $i_0(c) = (a, d)$, $i_0(d) = (1, b)$ extends to a unique group homomorphism $i_0 : \Xi_0 \to \Gamma_{-1} \times \Gamma_{-1}$ such that, for all $x \in \Gamma_0$, if $i_0(x) = (x_0, x_1)$, then $i_0(axa) = (x_1, x_0)$. By [6, Proposition 10], this induces, for all $n \geqq 1$, an injective group homomorphism: $i_n : \Xi_n \to \Gamma_{n-1} \times \Gamma_{n-1}$.

Proof of Proposition 5. Let us proceed by induction on n. We start with the essential case when $n = 0$, worked out in [6, Lemma 11]. Write $\Gamma_0 = \langle a, b, d |\ a^2 = b^2 = d^2 = (bd)^2 = (ad)^4 = 1\rangle$ (this is a Coxeter group). Let H_0 be the normal subgroup generated by $(ab)^2$. Then, by an immediate verification, Γ_0/H_0 is isomorphic to the direct product of a cyclic group of order 2 and a dihedral group of order 8.

We claim that H_0 is free of rank 3. Let L be the normal subgroup of Γ_0 generated by ab: then L has index 4 in Γ_0, contains H_0, and is shown, in the proof of [6, Lemma 11], to be isomorphic to $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})$.

Accordingly, by Kurosh's Theorem, if H_0 were not free, then it would contain a conjugate of ab, but this is not the case. Actually H_0 is contained in a subgroup of index 8 in Γ_0, free of rank 2 (see the proof of [6, Lemma 11]), hence has rank 3.

Now, for $n \geqq 1$, let us suppose that Γ_{n-1} has a normal subgroup H_{n-1} of index $2^{\alpha_{n-1}}$, which embeds as a subgroup of index $2^{\beta_{n-1}}$ in a direct product of 2^{n-1} non-abelian free groups of rank 3.

The homomorphism i_n described above embeds Ξ_n as a subgroup of index 8 in $\Gamma_{n-1} \times \Gamma_{n-1}$. Define $H'_n = i_n^{-1}(H_{n-1} \times H_{n-1})$: this a normal subgroup of index 2^k in Ξ_n, with $k \leqq 2\alpha_{n-1}$; so H'_n has index 2^{k+1} in Γ_n.

Then, using Lemma 7, $H_n = H'_n \cap {}^aH'_n$ has index 2^{α_n} in G, for some $\alpha_n \leqq 4\alpha_{n-1} + 1$. Combining the inclusions $H_n \subset H'_n \overset{\sim}{\to} H_{n-1} \times H_{n-1} \subset {F_3}^{2^{n-1}} \times {F_3}^{2^{n-1}}$, we obtain that H_n embeds as a subgroup of index 2^{β_n} in ${F_3}^{2^n}$, with $\beta_n = [H'_n : H_n] + 2[{F_3}^{2^{n-1}} : H_{n-1}] \leqq 2\alpha_{n-1} + 2\beta_{n-1}$.

Define $\alpha'_n = (13\cdot 4^n - 1)/3$ and $\beta'_n = (13\cdot 4^n - 15\cdot 2^n + 2)/3$. Then $\alpha'_0 = 4$, $\beta'_0 = 0$, and they satisfy, for all n: $\alpha'_n = 4\alpha'_{n-1} + 1$ and $\beta'_n = 2\alpha'_{n-1} + 2\beta'_{n-1}$. Therefore, an immediate induction gives $\alpha_n \leqq \alpha'_n$ and $\beta_n \leqq \beta'_n$ for all $n \geqq 0$. □

It is maybe worthwhile to restate the result avoiding reference to the particular sequence Γ_n.

Corollary 9. *For every finitely presented group G mapping onto the first Grigorchuk group Γ, there exist normal subgroups $N \subsetneq H$ in G, with H of finite 2-power index, such* *that H/N is isomorphic to a finite index subgroup in a direct product of non-abelian free groups.*

Proof. Let $p : G \to \Gamma$ be onto. Since G is finitely presented, p factors through Γ_n for some sufficiently large n, so that there exists a map p' from G onto Γ_n. Then take $N = \mathrm{Ker}(p')$ and $H = p'^{-1}(H_n)$. □

Combining Propositions 4 and 5, we also obtain the following statement:

Corollary 10. *In the group $\Gamma_{i(n)}$, the n-ball consists of $2^{i(n)+1}$-torsion elements, and there exists a normal subgroup of index $2^{\alpha_{i(n)}}$, which embeds in a direct product of free groups, with $\alpha_{i(n)} \leqq (13 \cdot n^{\log_\lambda(4)} - 1)/3$, and $\log_\lambda(4) \cong 6.60$.* □

Proof of Proposition 1. By Proposition 4, we may take i sufficiently large so that the n-ball of Γ_i consists of torsion elements. Since H_i is a finite index subgroup in a nontrivial direct product of free groups (see Proposition 5), it has infinite abelianization. There is a short exact sequence

$$1 \to H_i/[H_i, H_i] \to \Gamma_i/[H_i, H_i] \to \Gamma_i/H_i \to 1\,.$$

Accordingly, $G = \Gamma_i/[H_i, H_i]$ is an infinite, virtually abelian group, in which the n-ball consists of torsion elements. Moreover, since Γ_i/H_i is a finite 2-group, G is also solvable. This proves (i).

For (iii), take, instead, $G = \Gamma_i/[[H_i, H_i], [H_i, H_i]]$. Since H_i maps onto a non-abelian free group, its metabelianization is not virtually polycyclic, so that G is virtually metabelian, but not virtually polycyclic.

For (ii), take a morphism from H_i onto a polycyclic group W which is not virtually nilpotent, and let K be the kernel of this morphism. Since the normalizer of K has finite index in Γ_i, K has finitely many conjugates $K_1, \dots, K_\ell$. Set $L = \bigcap_{j=1}^{\ell} K_i$. Then the diagonal map $H_i/L \to \prod_{j=1}^{\ell} H_i/K_i$ is injective, hence embeds H_i/L in W^ℓ. On the other hand, observe that H_i/L projects onto W, so is not virtually nilpotent. It follows that $G = \Gamma_i/L$ is polycyclic but not virtually nilpotent. If W has been chosen metabelian, then we also have that G is virtually metabelian. □

Proof of Proposition 3. Keep the last construction in the previous proof. Then $h(\Gamma_i/L) \leqq h(\Gamma_i)$. Moreover, $h(\Gamma_i/L) > 0$ since Γ_i/L is solvable but not virtually nilpotent [10]. On the other hand, it is proved in [6] that $h(\Gamma_i) \to 0$. Thus we can obtain $h(G)$ arbitrarily small. □

Remark 11. Consider for every i an infinite quotient Q_i of Γ_i. In the topology of marked groups (defined in [5]; see also, for instance, [3]), the sequence (Q_i) converges to the Grigorchuk group Γ. Indeed, otherwise by compactness it would have another cluster point, which would be a proper quotient of Γ, and therefore would be finite. This is a contradiction since the infinite groups form a closed subset in the topology of marked groups.

Appendix. We gather here the technical results concerning torsion in the groups Γ_n. They are slight modifications of results in the papers [6] and [1].

Recall that $\lambda \cong 1.25$ denotes the real root of the polynomial $2X^3 - X^2 - X - 1$. We introduce on Γ_{-1} (and hence on all of its quotients) the metric $|\cdot|$ of [1]: it is defined by attributing a suitable weight to each of the generators a, b, c, d: $|a| = 2(\lambda - 1) \cong 0.47$, $|b| = 1 - |a| = \lambda^{-3} \cong 0.53$, $|c| = 2\lambda^2 - 3\lambda + 1 \cong 0.34$, and $|d| = -2\lambda^2 + \lambda + 2 \cong 0.19$. *Throughout this appendix the balls and the lengths are meant in the sense of this weighted metric.*

To check that the length of a, b, c, d is exactly the weight we have imposed, it suffices to check this in the abelianization of Γ_{-1}, the $\mathbb{F}_2$-vector space with basis (a, b, d) (which is also the abelianization of all Γ_n). There, it is a straightforward verification that the mapping $|\cdot|$ just defined extends to a length function by setting $|a\xi| = |a| + |\xi|$ for all $\xi \in \{b, c, d\}$.

Observe that if $\xi \in \{b, c, d\}$, and $i_n(\xi) = (\xi_0, \xi_1)$, we have

$$|\xi_0| + |\xi_1| = \lambda^{-1}(|\xi| + |a|). \tag{1}$$

Lemma 12. *Let $x \in \Gamma_0$ be any element. Set $x' = x$ if $x \in \Xi_0$ and $x' = xa$ otherwise; and set $i_0(x') = (x_0, x_1)$.*

Then $|x_0| + |x_1| \leqq \lambda^{-1}(|x| + |a|)$.

Suppose moreover that x is of minimal length among its conjugates, and that $x \notin \{b, c, d\}$.

Then $|x_0| + |x_1| \leqq \lambda^{-1}|x|$.

Proof. Fix $x \in \Gamma_0$, and let w be a word in the letters $\{a, b, c, d\}$, of minimal length[1)], representing x. Since every element in $\{b, c, d\}$ is the product of the two others, w can be chosen so that no two consecutive letters are in $\{b, c, d\}$.

Suppose now that x is of minimal length within its conjugacy class and that w is not a single letter. Maybe conjugating x by the last letter of w, we can suppose that w ends with the letter a. The minimality assumption then implies that w begins with a letter in $\{b, c, d\}$.

First case. $x \in \Xi_0$. Write $w = \xi^1(a\xi^2 a) \dots \xi^{2n-1}(a\xi^{2n}a)$, where $\xi^i \in \{b, c, d\}$ for $i = 1, \dots, 2n$. Write $i(x) = (x_0, x_1)$ and $i(\xi^i) = (\xi^i_0, \xi^i_1)$, so that $i(\xi^i) = (\xi^i_1, \xi^i_0)$. Then

$$\begin{aligned} |x_0| + |x_1| &\leqq (|\xi^1_0| + |\xi^2_1| + \dots + |\xi^{2n-1}_0| + |\xi^{2n}_1|) \\ &\quad + (|\xi^1_1| + |\xi^2_0| + \dots + |\xi^{2n-1}_1| + |\xi^{2n}_0|) \\ &= (|\xi^1_0| + |\xi^1_1|) + (|\xi^2_0| + |\xi^2_1|) + \dots + (|\xi^{2n}_0| + |\xi^{2n}_1|). \end{aligned}$$

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1) If $w = u_1 \dots u_n$, the length of w is defined as $|u_1| + \dots + |u_n|$.

By (1), we get

$$|x_0| + |x_1| \leqq \lambda^{-1} \sum_{i=1}^{2n} (|\xi^i| + |a|).$$

On the other hand, $|x| = \sum\limits_{i=1}^{2n} (|\xi^i| + |a|)$, so that finally $|x_0| + |x_1| \leqq \lambda^{-1}|x|$.

S e c o n d c a s e. $x \notin \Xi_0$, so that $xa \in \Xi_0$. Write $w = \xi^1(a\xi^2 a)\dots\xi^{2n-1}(a\xi^{2n}a)$ $\xi^{2n+1}a$, so that $\xi^1(a\xi^2 a)\dots\xi^{2n-1}(a\xi^{2n}a)\xi^{2n+1}$ represents xa in Γ_0. Write $i_0(xa) = (x_0, x_1)$, and $i_0(\xi^i) = (\xi^i_0, \xi^i_1)$. Then

$$\begin{aligned} |x_0| + |x_1| &\leqq (|\xi_0^1| + |\xi_1^2| + \dots + |\xi_0^{2n-1}| + |\xi_1^{2n}| + |\xi_0^{2n+1}|) \\ &\quad + (|\xi_1^1| + |\xi_0^2| + \dots + |\xi_1^{2n-1}| + |\xi_0^{2n}| + |\xi_1^{2n+1}|) \\ &= (|\xi_0^1| + |\xi_1^1|) + (|\xi_0^2| + |\xi_1^2|) + \dots + (|\xi_0^{2n+1}| + |\xi_1^{2n+1}|) \\ &= \lambda^{-1}\left(\sum_{i=1}^{2n+1} (|\xi^i| + |a|)\right) \quad \text{again by (1).} \end{aligned}$$

Since $|x| = \sum\limits_{i=1}^{2n+1} (|\xi^i| + |a|)$, we get $|x_0| + |x_1| \leqq \lambda^{-1}|x|$.

The other inequality $|x_0| + |x_1| \leqq \lambda^{-1}(|x| + |a|)$ is proved similarly: we must deal with the following cases:

- w begins and ends with the letter a: considering whether or not $x \in \Xi_0$, in both cases we obtain the stronger inequality $|x_0| + |x_1| \leqq \lambda^{-1}(|x| - |a|)$.
- w begins and ends with letters in $\{b, c, d\}$: considering whether or not $x \in \Xi_0$, in both cases we obtain the inequality $|x_0| + |x_1| \leqq \lambda^{-1}(|x| + |a|)$.
- w begins with the letter a and ends with a letter in $\{b, c, d\}$: in this case, replacing x by x^{-1} and w by w^{-1} (this is the word w read from right to left — recall that the generators are involutions), we reduce to the case, already carried out, in which w begins with a letter in $\{b, c, d\}$ and ends with the letter a, obtaining the inequality $|x_0| + |x_1| \leqq \lambda^{-1}|x|$.

Since the verifications are similar to the computations above and since we do not use this case in the sequel, we omit the details. □

Lemma 13. *For every $n \geqq -1$, and every element in the open λ^{n-1}-ball of Γ_{-1}, its image in Γ_n is of 2^{n+1}-torsion.*

P r o o f. For $n = -1$ we have $\lambda^{-2} = |a| + |d| \cong 0.66$, and the elements in the open λ^{-2}-ball are $1, a, b, c$, and d, and are of 2-torsion in Γ_{-1}.

For $n = 0$ we have $\lambda^{-1} = |a| + |c| \cong 0.81$, and the elements in the open λ^{-1}-ball are, besides the elements in the open λ^{-2}-ball already quoted, ad and its inverse da, which are of 4-torsion in Γ_0.

We can start an induction, and suppose that, for some $n \geqq 1$, we have already proved that for every element in the open λ^{n-2}-ball of Γ_{-1}, its image in Γ_{n-1} is of 2^n-torsion. Pick x in the open λ^{n-1}-ball of Γ_{-1}. We want to show that $x^{2^{n+1}} = 1$. We can suppose that x is of minimal length among its conjugates, and that $x \notin \{b, c, d\}$. Define x' as in Lemma 3, i.e., $\{x'\} = \{x, xa\} \cap \Xi_{-1}$. Denote by $[\cdot]$ the projection of Γ_{-1} onto Γ_0. Set $i_0([x']) = (x_0, x_1)$.

F i r s t c a s e. $x \in \Xi_{-1}$, i.e., $x = x'$. By Lemma 12, we have $|x_i| \leqq |x_0| + |x_1| \leqq \lambda^{-1}|x| \leqq \lambda^{n-2}$ for all $i \in \{0, 1\}$. By the induction hypothesis, x_0 and x_1 are of 2^n-torsion in Γ_{n-1}. Since i_0 induces an injection of Ξ_n into $\Gamma_{n-1} \times \Gamma_{n-1}$, this implies that x is of 2^n-torsion, hence of 2^{n+1}-torsion in Γ_n.

S e c o n d c a s e. $x \notin \Xi_{-1}$, i.e., $x' = xa$. Then $x^2 \in \Xi_{-1}$, and

$$i_0([x^2]) = i_0([xaaxaa]) = i_0([xa])i_0([a(xa)a]) = (x_0x_1, x_1x_0),$$

which is conjugate to (x_0x_1, x_0x_1) in $\Gamma_{-1} \times \Gamma_{-1}$. By Lemma 12, we have $|x_0x_1| \leqq |x_0| + |x_1| \leqq \lambda^{-1}|x| \leqq \lambda^{n-2}$. By the induction hypothesis, x_0x_1 is of 2^n-torsion in Γ_{n-1}, so $i_0([x^2])$ is of 2^n-torsion in $\Gamma_{n-1} \times \Gamma_{n-1}$. Since i_0 induces an injection of Ξ_n into $\Gamma_{n-1} \times \Gamma_{n-1}$, this implies that x^2 is of 2^n-torsion in $\Xi_n \subset \Gamma_n$, hence x is of 2^{n+1}-torsion in Γ_n. $\square$

P r o o f o f P r o p o s i t i o n 4. Suppose that x has word length $\leqq n$. Then

$$|x| \leqq |b|n = \lambda^{-3}n = \lambda^{\log_\lambda(n)-3} < \lambda^{i(n)-1}.$$

By Lemma 13, the image of x in $\Gamma_{i(n)}$ is of $2^{i(n)+1}$-torsion. $\square$

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